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Opial inequalities for a conformable Δ-fractional calculus on time scales

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Abstract. In this paper, an Opial-type inequality is introduced on time scale for a conformal Δ-fractional differentiable function of order α, $\alpha \in (0,1]$. In the case where the certain weight functions are included, one generalization of the Opial inequality is proved using conformal Δ-fractional calculus on time scales. Moreover, for n times conformal Δ-fractional differentiable function on time scale, $n \in \mathbb{N}$, an Opial inequality is obtained. In particular, through examples, the main results from the paper are compared with classical ones on generalized time scales.

At the end of the paper, we indicate possible applications of the obtained Opial-type inequalities in the consideration of stochastic dynamical equations where conformal Δ-fractional calculus on time scales is included, which requires further research.

1. Introduction

A time scale is any nonempty closed subset of the real line. The theory of time scales is a fairly new area of research. It was introduced in Stefan Hilger's Ph.D. thesis ([14]) as a way to unify the seemingly disparate fields of difference equations and differential equations. We begin by giving the basic calculus of time scales (see [3, 5]).

Let $\mathbb{T}$ be a nonempty closed subset of the real line.

Definition 1. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} \mid s < t\}$.

If $\sigma(t) > t$, we say that t is right-scattered, whereas if $\rho(t) < t$, we say that t is left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. The graininess functions $\mu, \nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$.

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Definition 2. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 4. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^k$. If there exists a number $a \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - a\,(\sigma(t) - s)\,| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be Δ-differentiable at t and we call a the Δ-derivative of f at t and denote it by $f^\Delta(t)$.

Theorem 1. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$. Then:*

1. *If f is differentiable at t, then f is continuous at t;*
2. *If f is continuous at t and t is right scattered, then f is Δ - differentiable at t, with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Theorem 2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^k$. Then the following statements are valid.*

1. *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

2. *For any constant c, $cf : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with*

$$(cf)^\Delta(t) = cf^\Delta(t).$$

Definition 5. A function $F : \mathbb{T} \to \mathbb{R}$ is called Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then Δ-integral of f on $[a, b]_\mathbb{T}$ is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad a, b \in \mathbb{T}.$$

One of the main subjects of the qualitative analysis on time scales is to prove dynamic inequalities. The content of this paper is motivated by some basic dynamic inequalities given in the articles [4, 13]. In particular, Bohner and Kaymakcalan in [8] introduced a dynamic Opial inequality on time scales, proving that

$$\int_0^h |(f + f^\sigma)f^\Delta|(t)\Delta t \leq h \int_0^h |f^\Delta|^2(t)\Delta t, \tag{1}$$

where $0, h \in \mathbb{T}$, $h > 0$, $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ is Δ-differentiable with $f(0) = 0$ and $f^\sigma(t) = f(\sigma(t))$. The dynamic Opial inequality (1) contains both the classical continuous Opial inequality ([8], Theorem 1.1) and the classical discrete Opial inequality ([8], Theorem 1.2) as special cases.

On the other hand, a conformable fractional calculus on an arbitrary time scale is a natural extension of the conformable fractional calculus (for example, see [7, 18]). As the fractional calculus always attracted interest of researchers due to its numerous applications in many fields (for example, see the references cited within [18]), we were interested to study Opial-type inequalities using conformable Δ-fractional calculus on time scales.

This paper is organized as follows. The second section is dedicated to the basic notions of conformable Δ-fractional derivative and integral on time scales. In the third section, we present the main results through three theorems. The first theorem (Theorem 8) introduces an Opial inequality on time scale for a conformal Δ-fractional differentiable function of order α, $\alpha \in (0, 1]$, and the second (Theorem 9), using conformable Δ-fractional calculus, proves one generalization of the Opial inequality on time scale when the certain weight functions are included. The third theorem (Theorem 10) is dedicated to an Opial inequality for n times conformal Δ-fractional differentiable function on time scale, where $n \in \mathbb{N}$. Through examples, the main results from this paper are compared with classical ones on generalized time scales. In conclusions, we indicate possible applications of the obtained Opial-type inequalities in the consideration of stochastic dynamical equations, where conformal Δ-fractional calculus on time scale is included, which requires further research.

2. Preliminaries

Fractional calculus is nowadays one of the most intensively developing areas of mathematical analysis (see [1,15,17]). We present the basic concepts of conformable Δ-fractional derivative and integral on time scales, which were introduced in [18].

In the following, we assume that $\alpha \in (0, 1]$ unless it is emphasized that a non-integer value of α, $\alpha > 0$, belongs to some other interval.

Definition 6. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $T_\alpha(f^\Delta)(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there exists a neighborhood U of t such that

$$|(f(\sigma(t)) - f(s))(\sigma(t))^{1-\alpha} - T_\alpha(f^\Delta)(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $T_\alpha(f^\Delta)(t)$ the conformable delta (Δ) fractional derivative of f of order α at t. Moreover, we say that f is conformable Δ-fractional differentiable of order α on $\mathbb{T}^k$ provided $T_\alpha(f^\Delta)(t)$ exists for all $t \in \mathbb{T}^k$. The function $T_\alpha(f^\Delta)(t) : \mathbb{T}^k \to \mathbb{R}$ is then called the conformable Δ-fractional derivative of f of order α on T^k. We define the conformable Δ-fractional derivative at 0 as

$$T_\alpha(f^\Delta)(0) = \lim_{t\to 0} T_\alpha(f^\Delta)(t).$$

Theorem 3. *Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}^k$ and $\alpha \in (0,1]$. Then we have the following.*

1. *If f is conformable Δ-fractional differentiable of order α at t, then f is continuous at t.*
2. *If f is continuous at t and t is right-scattered, then f is conformable Δ-fractional differentiable of order α at t with*
$$T_\alpha(f^\Delta)(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}(\sigma(t))^{1-\alpha}.$$
3. *If t is right-dense, then f is conformable Δ-fractional differentiable of order α at t if and only if the limit*
$$\lim_{s\to t} \frac{f(t)-f(s)}{t-s} t^{1-\alpha}$$
exists as a finite number. In this case,
$$T_\alpha(f^\Delta)(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s} t^{1-\alpha}.$$
4. *If f is conformable Δ-fractional differentiable of order α at t, then*
$$f(\sigma(t)) = f(t) + \mu(t)T_\alpha(f^\Delta)(t)(\sigma(t))^{\alpha-1}.$$

Theorem 4. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are conformable Δ-fractional differentiable of order α at $t \in \mathbb{T}^k$. Then the following statements are valid.*

1. *For all constants λ_1, λ_2, the sum $\lambda_1 f + \lambda_2 g : \mathbb{T} \to \mathbb{R}$ is conformable Δ-fractional differentiable of order α at $t \in \mathbb{T}^k$ with*
$$T_\alpha((\lambda_1 f + \lambda_2 g)^\Delta)(t) = \lambda_1 T_\alpha(f^\Delta)(t) + \lambda_2 T_\alpha(g^\Delta)(t).$$
2. *The product $fg : \mathbb{T} \to \mathbb{R}$ is conformable Δ-fractional differentiable of order α at t with*
$$\begin{aligned} T_\alpha((fg)^\Delta)(t) &= T_\alpha(f^\Delta)(t)g(t) + f(\sigma(t))T_\alpha(g^\Delta)(t) \\ &= f(t)T_\alpha(g^\Delta)(t) + T_\alpha(f^\Delta)(t)g(\sigma(t)). \end{aligned}$$

Definition 7. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the indefinite α-conformable Δ-fractional integral of f by
$$I_\alpha(f^\Delta)(t) + C = \int f(t)\Delta_\alpha t = \int f(t)(\sigma(t))^{\alpha-1}\Delta t,$$
where C is an arbitrary constant. $I_\alpha(f^\Delta)(t)$ is called a pre-antiderivative of f. We define the Cauchy α-conformable Δ-fractional integral by
$$\int_a^b f(t)\Delta_\alpha t = I_\alpha(f^\Delta)(b) - I_\alpha(f^\Delta)(a),$$
for all $a, b \in \mathbb{T}$. A function $I_\alpha(f^\Delta) : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $(T_\alpha I_\alpha(f^\Delta))(t) = f(t)$, for all $t \in \mathbb{T}^k$.

Theorem 5 (Existence of antiderivatives)**.** *For every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, there exists a function $I_\alpha(f^\Delta)$ such that*

$$(T_\alpha I_\alpha(f^\Delta))(t) = f(t).$$

Theorem 6. *Let $a, b, c \in \mathbb{T}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f, g : \mathbb{T} \to \mathbb{R}$ be rd-continuous functions. Then*

1. $\displaystyle\int_a^b (\lambda_1 f(t) + \lambda_2 g(t))\Delta_\alpha t = \lambda_1 \int_b^a f(t)\Delta_\alpha t + \lambda_2 \int_b^a g(t)\Delta_\alpha t;$

2. $\displaystyle\int_a^b f(t)\Delta_\alpha t = -\int_b^a f(t)\Delta_\alpha t;$

3. $\displaystyle\int_a^b f(\sigma(t))T_\alpha(g^\Delta)(t)\Delta_\alpha t = f(b)g(b) - f(a)g(a) - \int_a^b T_\alpha(f^\Delta)(t)g(t)\Delta_\alpha t;$

4. $\displaystyle\int_a^b f(t)T_\alpha(g^\Delta)(t)\Delta_\alpha t = f(b)g(b) - f(a)g(a) - \int_a^b T_\alpha(f^\Delta)(t)g(\sigma(t))\Delta_\alpha t;$

5. $\displaystyle\int_a^a f(t)\Delta_\alpha t = 0.$

Definition 8. Let $\mathbb{T}$ be a time scale, $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, and f is n times Δ-differentiable at $t \in \underbrace{\mathbb{T}^k \times \cdots \times \mathbb{T}^k}_{n}$. We define the conformable Δ-fractional derivative of f of order α as

$$T_\alpha(f^\Delta)(t) = T_{\alpha-n}(f^{\Delta^{n+1}})(t).$$

Theorem 7. *Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}$. The following relation is valid*

$$T_{\alpha-n}(f^{\Delta^{n+1}})(t) = (\sigma(t))^{n+1-\alpha} f^{\Delta^{n+1}}(t).$$

3. Main results

For an absolutely continuous function $f : [0, h] \to \mathbb{R}$ with $f(0) = 0$, we have classical continuous Opial inequality (classical Opial integral inequality)

$$\int_0^h |f(t)f'(t)|dt \le \frac{h}{2} \int_0^h |f'(t)|^2 dt,$$

where the equality holds in the case that $f(t) = ct$ for some constant c (for example, see [8], Theorem 1.4.1).

Remark 1. Considering that in the time scales calculus,

$$(f^2(t))^\Delta = ((f + f^\sigma)f^\Delta)(t),$$

it follows that for $\mathbb{T} = \mathbb{R}$, since

$$(f^2(t))' = 2f(t)f'(t),$$

the left-hand side of the dynamic Opial inequality (1) transforms into the left-hand side of the classical Opial integral inequality multiplied by two.

In this paper, we consider an α-conformable Δ-fractional integral

$$\int_0^h |f(t)T_\alpha(f^\Delta)(t)|\Delta_\alpha t$$

which for $\alpha = 1$, when $T_\alpha(f^\Delta)(t) = f^\Delta(t)$ and $\Delta_\alpha t = \Delta t$, becomes Δ-integral of the form

$$\int_0^h |f(t)(f^\Delta)(t)|\Delta t.$$

Let $\mathbb{T}$ be a time scale such that $0, h \in \mathbb{T}$ and $h > 0$.

Theorem 8. *Let $\alpha \in (0,1]$ and $f : [0,h] \cap \mathbb{T} \to \mathbb{R}$ be conformable Δ-fractional differentiable function of order α on $\mathbb{T}^k$. Then*

$$\int_0^h |f(t)T_\alpha(f^\Delta)(t)|\Delta_\alpha t \le \frac{\gamma}{2\alpha}\int_0^h |T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t + \beta\int_0^h |T_\alpha(f^\Delta)(t)|\Delta_\alpha t,$$

where

$$\gamma = \min_{\tau\in[0,h]\cap\mathbb{T}} \nu(\tau), \quad \nu(\tau) = \max\{\tau^\alpha,\, h^\alpha - \tau^\alpha\}, \quad \tau \in [0,h]\cap\mathbb{T},$$

and $\beta = \max\{|f(0)|, |f(h)|\}$.

Proof. Denote

$$y(t) = \int_0^t |T_\alpha(f^\Delta)(s)|\Delta_\alpha s, \quad z(t) = \int_t^h |T_\alpha(f^\Delta)(s)|\Delta_\alpha s.$$

Then $T_\alpha(y^\Delta)(t) = |T_\alpha(f^\Delta)(t)|$ and $T_\alpha(z^\Delta)(t) = -|T_\alpha(f^\Delta)(t)|$. Therefore, we get

$$\begin{aligned} |f(t)| &\le |f(t) - f(0)| + |f(0)| \\ &\le \int_0^t |T_\alpha(f^\Delta)(s)|\Delta_\alpha s + |f(0)| = y(t) + |f(0)|. \end{aligned} \tag{2}$$

In a similar way, we come to the inequality

$$|f(t)| \le \int_t^h |T_\alpha(f^\Delta)(s)|\Delta_\alpha s + |f(h)| = z(t) + |f(h)|.$$

Let $\tau \in [0, h] \cap \mathbb{T}$. Relying on (2), we obtain

$$\begin{aligned}\int_0^\tau |f(t)||T_\alpha(f^\Delta)(t)|\Delta_\alpha t &\leq \int_0^\tau [y(t) + |f(0)|]T_\alpha(y^\Delta)(t)\Delta_\alpha t \\ &= \int_0^\tau y(t)T_\alpha(y^\Delta)(t)\Delta_\alpha t + |f(0)| \int_0^\tau T_\alpha(y^\Delta)(t)\Delta_\alpha t.\end{aligned}$$

On the other hand, the item 4 from Theorem 6 indicates that the first integral on the right-hand side in the last inequality takes the form

$$\int_0^\tau y(t)T_\alpha(y^\Delta)(t)\Delta_\alpha t = y^2(\tau) - y^2(0) - \int_0^\tau T_\alpha(y^\Delta)(t)y(\sigma(t))\Delta_\alpha t,$$

and since $y(\sigma(t)) \geq y(t)$, we obtain

$$\int_0^\tau y(t)T_\alpha(y^\Delta)(t)\Delta_\alpha t \leq y^2(\tau) - y^2(0) - \int_0^\tau T_\alpha(y^\Delta)(t)y(t)\Delta_\alpha t \leq \frac{y^2(\tau)}{2}.$$

Therefore, we have

$$\begin{aligned}\int_0^\tau |f(t)||T_\alpha(f^\Delta)(t)|\Delta_\alpha t &\leq \frac{y^2(\tau)}{2} + |f(0)|y(\tau) \\ &= \frac{1}{2}\Big[\int_0^\tau |T_\alpha(f^\Delta)(t)|\Delta_\alpha t\Big]^2 + |f(0)| \int_0^\tau |T_\alpha(f^\Delta)(t)|\Delta_\alpha t.\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the first integral in the last row, we obtain

$$\begin{aligned}\Big[\int_0^\tau |T_\alpha(f^\Delta)(t)|\Delta_\alpha t\Big]^2 &\leq \Big[\Big(\int_0^\tau \Delta_\alpha t\Big)^{\frac{1}{2}}\Big(\int_0^\tau |T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t\Big)^{\frac{1}{2}}\Big]^2 \\ &= \int_0^\tau (\sigma(t))^{\alpha-1}\Delta t \int_0^\tau |T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t.\end{aligned}$$

Using the expression

$$(x^\gamma(t))^\Delta = \gamma\Big[\int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{\gamma-1}dh\Big]x^\Delta(t), \tag{3}$$

derivable from the Theorem 1.90 (Chain Rule) given in [9], and the fact that $t \leq \sigma(t)$, we find

$$\begin{aligned}(t^\alpha)^\Delta &= \alpha \int_0^1 [h\sigma(t) + (1-h)t]^{\alpha-1}dh \\ &\leq \alpha \int_0^1 [h\sigma(t) + (1-h)\sigma(t)]^{\alpha-1}dh \\ &= \alpha(\sigma(t))^{\alpha-1}.\end{aligned}$$

So, replacing $(\sigma(t))^{\alpha-1}$ with $\left(\frac{t^\alpha}{\alpha}\right)^\Delta$, we get

$$\begin{aligned}
&\int_0^\tau |f(t)||T_\alpha(f^\Delta)(t)|\Delta_\alpha t \\
&\leq \frac{1}{2}\int_0^\tau \left(\frac{t^\alpha}{\alpha}\right)^\Delta \Delta t \int_0^\tau |T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t + |f(0)|\int_0^\tau |T_\alpha(f^\Delta)(t)|\Delta_\alpha t \\
&= \frac{\tau^\alpha}{2\alpha}\int_0^\tau |T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t + |f(0)|\int_0^\tau |T_\alpha(f^\Delta)(t)|\Delta_\alpha t.
\end{aligned} \tag{4}$$

Similarly, one can show

$$\begin{aligned}
&\int_\tau^h |f(t)||T_\alpha(f^\Delta)(t)|\Delta_\alpha t \\
&\leq \frac{h^\alpha-\tau^\alpha}{2\alpha}\int_\tau^h |T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t + |f(h)|\int_\tau^h |T_\alpha(f^\Delta)(t)|\Delta_\alpha t.
\end{aligned} \tag{5}$$

By putting $\nu(\tau) = \max\{\tau^\alpha,\, h^\alpha - \tau^\alpha\}$, $\tau \in [0,h]\cap\mathbb{T}$, $\beta = \max\{|f(0)|,\, |f(h)|\}$ and adding inequalities (4) and (5), we obtain

$$\int_0^h |f(t)T_\alpha(f^\Delta)(t)|\Delta_\alpha t \leq \frac{\nu(\tau)}{2\alpha}\int_0^h |T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t + \beta\int_0^h |T_\alpha(f^\Delta)(t)|\Delta_\alpha t.$$

The last inequality is true for any $\tau \in [0,h]\cap\mathbb{T}$. Therefore, it is also true if $\nu(\tau)$ is replaced by $\gamma = \min_{\tau\in[0,h]\cap\mathbb{T}}\nu(\tau)$ and the proof is complete. □

Example 1. For $\alpha = 1$ and Δ-differentiable function $f : [0,h]\cap\mathbb{T} \to \mathbb{R}$, the inequality from Theorem 8 becomes

$$\int_0^h |f(t)(f^\Delta)(t)|\Delta t \leq \frac{1}{2}\left\{\gamma\int_0^h |(f^\Delta)(t)|^2\Delta t + 2\beta\int_0^h |(f^\Delta)(t)|\Delta t\right\},$$

where $\gamma = \min_{\tau\in[0,h]\cap\mathbb{T}}\nu(\tau)$, $\nu(\tau) = \max\{\tau, h-\tau\}$, $\tau \in [0,h]\cap\mathbb{T}$, and $\beta = \max\{|f(0)|, |f(h)|\}$.
Therefore, for $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$ we get one type of the continuous Opial inequality

$$\int_0^h |f(t)f'(t)|dt \leq \frac{1}{2}\left\{\gamma\int_0^h |(f'(t)|^2 dt + 2\beta\int_0^h |f'(t)|dt\right\},$$

and for $\alpha = 1$ and $\mathbb{T} = \mathbb{Z}$ we obtain one type of the discrete Opial inequality

$$\begin{aligned}
&\sum_{t=0}^{h-1}|f(t)(f(t+1)-f(t))| \\
&\leq \frac{1}{2}\left\{\gamma\sum_{t=0}^{h-1}|f(t+1)-f(t)|^2 + 2\beta\sum_{t=0}^{h-1}|f(t+1)-f(t)|\right\},
\end{aligned}$$

where $h \in \mathbb{Z}$ and $h > 0$.

Theorem 9. *Let $\alpha \in (0,1]$, and p and q be positive and continuous functions on $[0,h]$ such that $\int_0^h \Delta_\alpha t/p(t) < \infty$ and q non-increasing. Then, for conformable Δ-fractional differentiable function $f : [0,h] \cap \mathbb{T} \to \mathbb{R}$ of order α with $f(0) = 0$, we have*

$$\int_0^h q^\sigma(t)|f(t)||T_\alpha(f^\Delta)(t)|\Delta_\alpha t$$
$$\leq \frac{1}{2}\left\{\int_0^h \frac{\Delta_\alpha t}{p(t)}\right\}\left\{\int_0^h p(t)q(t)|T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t\right\},$$

where $q^\sigma(t) = q(\sigma(t))$.

Proof. Let us consider the function $g(t)$ defined by the integral

$$g(t) = \int_0^t \sqrt{q^\sigma(s)}|T_\alpha(f^\Delta)(s)|\Delta_\alpha s.$$

Hence we have $T_\alpha(g^\Delta)(t) = \sqrt{q^\sigma(t)}|T_\alpha(f^\Delta)(t)|$, and taking into account that q is a non-increasing function, we get

$$|f(t)| \leq \int_0^t |T_\alpha(f^\Delta)(s)|\Delta_\alpha s$$
$$\leq \int_0^t \sqrt{\frac{q^\sigma(s)}{q(t)}}|T_\alpha(f^\Delta)(s)|\Delta_\alpha s = \frac{g(t)}{\sqrt{q(t)}}.$$

Therefore,

$$\int_0^h q^\sigma(t)|f(t)||T_\alpha(f^\Delta)(t)|\Delta_\alpha t \leq \int_0^h q^\sigma(t)\frac{g(t)}{\sqrt{q(t)}\sqrt{q^\sigma(t)}}T_\alpha(g^\Delta)(t)\Delta_\alpha t$$
$$\leq \int_0^h g(t)T_\alpha(g^\Delta)(t)\Delta_\alpha t \leq \frac{g^2(h) - g^2(0)}{2}.$$

Finally, using the time scales of Cauchy-Schwarz inequality, we obtain at the following inequality

$$\frac{g^2(h)}{2} = \frac{1}{2}\Big[\int_0^h \frac{1}{\sqrt{p(t)}}\sqrt{p(t)q^\sigma(t)}|T_\alpha(f^\Delta)(t)|\Delta_\alpha t\Big]^2$$
$$\leq \frac{1}{2}\int_0^h \frac{\Delta_\alpha t}{p(t)}\int_0^h p(t)q(t)|T_\alpha(f^\Delta)(t)|^2\Delta_\alpha t,$$

which completes the proof. □

Remark 2. In Theorem 9, p and q are continuous functions on the nonempty closed subset $[0,h]$ of the real line $\mathbb{R}$, which means that p and q are *rd*-continuous on $[0,h]$ (see [9], Theorem 1.60). Consequently, based on Theorem 5, the functions p and q have antiderivatives.

Example 2. For $\alpha = 1$ and Δ-differentiable function $f : [0,h]\cap\mathbb{T} \to \mathbb{R}$ with $f(0) = 0$, the inequality from Theorem 9 becomes

$$\int_0^h q^\sigma(t)|f(t)f^\Delta(t)|\Delta t \leq \frac{1}{2}\left\{\int_0^h \frac{\Delta t}{p(t)}\right\}\left\{\int_0^h p(t)q(t)|(f^\Delta)(t)|^2\Delta t\right\},$$

where p and q are positive and continuous functions on $[0,h]$ such that $\int_0^h \Delta t/p(t) < \infty$ and q non-increasing. Therefore, for $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$ we get one generalization of the continuous Opial inequality

$$\int_0^h q(t)|f(t)f'(t)|dt \leq \frac{1}{2}\left\{\int_0^h \frac{dt}{p(t)}\right\}\left\{\int_0^h p(t)q(t)|f'(t)|^2\Delta t\right\},$$

(see [2], Theorem 2.5.1), and for $\alpha = 1$ and $\mathbb{T} = \mathbb{Z}$, we obtain one generalization of the discrete Opial inequality,

$$\sum_{t=0}^{h-1} q(t+1)|f(t)(f(t+1)-f(t))|$$
$$\leq \frac{1}{2}\left\{\sum_{t=0}^{h-1}\frac{1}{p(t)}\right\}\left\{\sum_{t=0}^{h-1}p(t)q(t)|f(t+1)-f(t)|^2\right\},$$

where $h \in \mathbb{Z}$ and $h > 0$.

Theorem 10. *Suppose $\alpha \in (n-1,n]$ and $m, n \in \mathbb{N}$. Then, for n times conformable Δ-fractional differentiable function $f : [0,h]\cap\mathbb{T} \to \mathbb{R}$ with*

$$f(0) = T_{\alpha-n+1}(f^\Delta)(0) = \cdots = T_{\alpha-n+1}(f^{\Delta^{n-1}})(0) = 0,$$

we have

$$\int_0^h |f(t)|^m \left|T_{\alpha-n+1}(f^{\Delta^n})(t)\right|\Delta_\alpha t$$
$$\leq \frac{1}{m+1}\Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{mn}\int_0^h |T_{\alpha-n+1}(f^{\Delta^n})(t)|^{m+1}\Delta_\alpha t. \tag{6}$$

Proof. We define a function g by the multiple integral

$$g(t) = \int_0^t\int_0^{t_{n-1}}\cdots\int_0^{t_2}\int_0^{t_1}\left|T_{\alpha-n+1}(f^{\Delta^n})(s)\right|\Delta_\alpha s\Delta_\alpha t_1\cdots\Delta_\alpha t_{n-2}\Delta_\alpha t_{n-1}.$$

Hence, we have

$$T_{\alpha-n+1}(g^\Delta)(t) = \int_0^t\int_0^{t_{n-2}}\cdots$$
$$\cdots\int_0^{t_2}\int_0^{t_1}\left|T_{\alpha-n+1}(f^{\Delta^n})(s)\right|\Delta_\alpha s\Delta_\alpha t_1\cdots\Delta_\alpha t_{n-3}\Delta_\alpha t_{n-2},$$

$$T_{\alpha-n+1}(g^{\Delta^{n-1}})(t) = \int_0^t \left|T_{\alpha-n+1}(f^{\Delta^n})(s)\right|\Delta_\alpha s,$$

$$T_{\alpha-n+1}(g^{\Delta^n})(t) = \left|T_{\alpha-n+1}(f^{\Delta^n})(t)\right|.$$

For $0 \le t \le h$, we can see that

$$
\begin{aligned}
|f(t)| &\le \int_0^t |T_{\alpha-n+1}(f^{\Delta})(t_1)|\Delta_\alpha t_1 \\
&\le \int_0^t \int_0^{t_1} |T_{\alpha-n+1}(f^{\Delta^2})(t_2)|\Delta_\alpha t_2 \Delta_\alpha t_1 \\
&\le \cdots \le g(t).
\end{aligned} \tag{7}
$$

Further, we find

$$
\begin{aligned}
g(t) &= \int_0^t T_{\alpha-n+1}(g^{\Delta})(s)\Delta_\alpha s \le \int_0^t T_{\alpha-n+1}(g^{\Delta})(t)\Delta_\alpha s \\
&\le \int_0^h T_{\alpha-n+1}(g^{\Delta})(t)\Delta_\alpha s = T_{\alpha-n+1}(g^{\Delta})(t)\int_0^h (\sigma(s))^{\alpha-n}\Delta s \\
&= T_{\alpha-n+1}(g^{\Delta})(t)\int_0^h \Big(\frac{s^{\alpha-n+1}}{\alpha-n+1}\Big)^{\Delta}\Delta s = \frac{h^{\alpha-n+1}}{\alpha-n+1}T_{\alpha-n+1}(g^{\Delta})(t),
\end{aligned}
$$

and taking into account the inequality (7), we get

$$
\begin{aligned}
|f(t)| \le g(t) &\le \Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^2 T_{\alpha-n+1}(g^{\Delta^2})(t) \\
&\le \cdots \le \Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{n-1} T_{\alpha-n+1}(g^{\Delta^{n-1}})(t).
\end{aligned}
$$

The last inequality allows us to reach the following conclusion

$$
\begin{aligned}
&\int_0^h |f(t)|^m |T_{\alpha-n+1}(f^{\Delta^n})(t)|\Delta_\alpha t \\
&\le \int_0^h \Big(\Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{n-1} T_{\alpha-n+1}(g^{\Delta^{n-1}})(t)\Big)^m T_{\alpha-n+1}(g^{\Delta^n})(t)\Delta_\alpha t \\
&= \Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{m(n-1)} \int_0^h [T_{\alpha-n+1}(g^{\Delta^{n-1}})(t)]^m T_{\alpha-n+1}(g^{\Delta^n})(t)\Delta_\alpha t.
\end{aligned}
$$

For simplicity, let us denote $y(t) = T_{\alpha-n+1}(g^{\Delta^{n-1}})(t)$. Now we have

$$
\begin{aligned}
&\int_0^h |f(t)|^m |T_{\alpha-n+1}(f^{\Delta^n})(t)|\Delta_\alpha t \\
&\le \Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{m(n-1)} \int_0^h (y(t))^m T_{\alpha-n+1}(y^{\Delta})(t)\Delta_\alpha t \\
&= \Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{m(n-1)} \int_0^h (y(t))^m y^{\Delta}(t)\Delta t.
\end{aligned}
$$

By virtue of the chain rule (3), and the fact that $y^{\Delta}(t) > 0$, we conclude

$$
(y(t))^m y^{\Delta}(t) \le \frac{1}{m+1}(y^{m+1}(t))^{\Delta},
$$

and obtain the inequality

$$\int_0^h |f(t)|^m |T_{\alpha-n+1}(f^{\Delta^n})(t)|\Delta_\alpha t$$
$$\leq \frac{1}{m+1}\Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{m(n-1)} \int_0^h (y^{m+1}(t))^{\Delta}\Delta t.$$

However, the right-hand side integral becomes

$$\int_0^h (y^{m+1}(t))^{\Delta}\Delta t = (y^{m+1}(h)) = \Big(T_{\alpha-n+1}(g^{\Delta^{n-1}})(h)\Big)^{m+1}$$
$$= \Big(\int_0^h |T_{\alpha-n+1}(f^{\Delta^n})(t)|\Delta_\alpha t\Big)^{m+1}.$$

Applying Hölder's inequality with indices $\frac{m+1}{m}$ and $m+1$ to the last integral on the right side, we get

$$\int_0^h (y^{m+1}(t))^{\Delta}\Delta t$$
$$\leq \Big\{\Big(\int_0^h \Delta_\alpha t\Big)^{\frac{m}{m+1}}\Big(\int_0^h |T_{\alpha-n+1}(f^{\Delta^n})(t)|^{m+1}\Delta_\alpha t\Big)^{\frac{1}{m+1}}\Big\}^{m+1}$$
$$= \Big(\int_0^h (\sigma(t))^{\alpha-n}\Delta t\Big)^m \int_0^h |T_{\alpha-n+1}(f^{\Delta^n})(t)|^{m+1}\Delta_\alpha t.$$

After calculating the first integral in the last row, we come to the inequality

$$\int_0^h (y^{m+1}(t))^{\Delta}\Delta t \leq \Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^m \int_0^h |T_{\alpha-n+1}(f^{\Delta^n})(t)|^{m+1}\Delta_\alpha t.$$

Thus, we finally arrive at the inequality (6), i.e.

$$\int_0^h |f(t)|^m \Big|T_{\alpha-n+1}(f^{\Delta^n})(t)\Big|\Delta_\alpha t$$
$$\leq \frac{1}{m+1}\Big(\frac{h^{\alpha-n+1}}{\alpha-n+1}\Big)^{mn} \int_0^h |T_{\alpha-n+1}(f^{\Delta^n})(t)|^{m+1}\Delta_\alpha t,$$

which is to be proved. □

Corollary 1. *Suppose $\alpha \in (0,1]$ and $m \in \mathbb{N}$. For conformable Δ-fractional differentiable function $f : [0,h] \cap \mathbb{T} \to \mathbb{R}$ with $f(0) = 0$, we have*

$$\int_0^h |f(t)|^m \Big|T_\alpha(f^{\Delta})(t)\Big|\Delta_\alpha t \leq \frac{1}{m+1}\Big(\frac{h^\alpha}{\alpha}\Big)^m \int_0^h |T_\alpha(f^{\Delta})(t)|^{m+1}\Delta_\alpha t.$$

Proof. This is Theorem 10 with $n = 1$. □

Example 3. In Theorem 10, let us that $\alpha = n, n \in \mathbb{N}, n > 1$. Then, the n times conformable Δ-fractional differentiable function $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with

$$f(0) = T_{\alpha-n+1}(f^{\Delta})(0) = \cdots = T_{\alpha-n+1}(f^{\Delta^{n-1}})(0) = 0$$

becomes an n times Δ-differentiable function $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $f(0) = f^{\Delta}(0) = \cdots = f^{\Delta^{n-1}}(0) = 0$, and the inequality (6) from Theorem 10 takes the form

$$\int_0^h |f(t)|^m |f^{\Delta^n}(t)| \Delta t \le \frac{h^{mn}}{m+1} \int_0^h |f^{\Delta^n}(t)|^{m+1} \Delta t,$$

where $m \in \mathbb{N}$.

Consequently, for $\alpha = n$ and $\mathbb{T} = \mathbb{R}$, the last inequality becomes one generalization of the classical continuous Opial inequality involving the n-th $(n > 1)$ derivative of the given function f,

$$\int_0^h |f(t)|^m |f^{(n)}(t)| dt \le \frac{h^{mn}}{m+1} \int_0^h |f^{(n)}(t)|^{m+1} dt,$$

(see [2], Chapter 3). On the other hand, for $\alpha = n$ and $\mathbb{T} = \mathbb{Z}$, $h \in \mathbb{Z}$, $h > 0$, the inequality (6) from Theorem 10 becomes one generalization of the classical discrete Opial inequality,

$$\sum_{t=0}^{h-1} |f(t)|^m |\Delta^n f(t)| \le \frac{h^{mn}}{m+1} \sum_{t=0}^{h-1} |\Delta^n f(t)|^{m+1},$$

involving the forward difference operator of order n $(n > 1)$,

$$\Delta^n f(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(t+k).$$

In particular, for $\alpha = n = m = 1$ and Δ-differentiable function $f : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $f(0) = 0$, the inequality (6) from Theorem 10 becomes

$$\int_0^h |f(t) f^{\Delta}(t)| \Delta t \le \frac{h}{2} \int_0^h |f^{\Delta}(t)|^2 \Delta t.$$

Thus, for $\alpha = n = m = 1$ and $\mathbb{T} = \mathbb{R}$, the inequality (6) becomes the classical continuous Opial inequality,

$$\int_0^h |f(t) f'(t)| dt \le \frac{h}{2} \int_0^h |f'(t)|)^2 dt.$$

On the other hand, if $\alpha = n = m = 1$ and $\mathbb{T} = \mathbb{Z}$, then $h \in \mathbb{Z}$, $h > 0$, and the inequality (6) becomes the classical discrete Opial inequality,

$$\sum_{t=0}^{h-1} |f(t)(f(t+1) - f(t))| \le \frac{h}{2} \sum_{t=0}^{h-1} |f(t+1) - f(t)|^2.$$

4. Conclusions

The fractional calculus has numerous applications in many fields (engineering, economics and finance, signal processing, dynamics of earthquakes, geology, probability and statistics, chemical engineering, physics, thermodynamics, neural networks, etc.). Therefore, it is one of the most intensively developing areas of mathematical analysis nowadays.

Several definitions of fractional derivative have been proposed. A fractional derivative which satisfies the well-known formula for the derivative of the product (the quotient) of two functions and the chain rule, etc., is called the conformable fractional derivative ([18]).

On the other hand, the theory of time scales was introduced in order to unify continuous and discrete analysis. Consequently, a conformable fractional calculus on an arbitrary time scale is a natural extension of the conformable fractional calculus.

Opial inequalities and many of their generalizations have various applications in the theory of differential and difference equations ([2]). From there, the need to study the Opial inequalities on time scales naturally arose ([8,9]).

The discovered concept of time scales began to be applied to probability theory as well, thus unifying discrete, continuous, and many other cases (for example, see [6, 16]). Stochastic calculus on time scales and stochastic time scales, meaning the time scale is generated by sampling a random variable, are still in development (for example, see references within [6]).

In this paper, we proved some Opial-type inequalities using a conformable Δ-fractional calculus on time scales.

In stochastic analysis, it is well known that the time series analysis implies knowledge of stochastic difference equations, while the continuous stochastic processes involve the study of stochastic differential equations. Therefore, the possibility of studying stochastic dynamic equations on time scales has opened up ([10–12]). Consequently, the obtained Opial inequalities in this paper can find their application in the consideration of stochastic dynamical equations where conformal Δ-fractional calculus on time scales is included. However, such an application points to further investigations of stochastic dynamical equations using conformable Δ-fractional calculus on time scales.

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References

[1] T. Abdeljawad, *On conformable fractional calculus*, Journal of Computational and Applied Mathematics, 279 (4) (2015), 57–66.

[2] R. P. Agarwal and P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer, Dordrecht, 1995.

[3] R. Agarwal, M. Bohner, *Basic calculus on time scales and some of its applications*, Results in Mathematics, 35 (1-2) (1999), 3–22.

[4] R. Agarwal, M. Bohner, A. Peterson, *Inequalities on time scales: A survey*, Mathematical Inequalities and Applications, 4 (4) (2001), 535–557.

[5] R. Agarwal, M. Bohner, D. O'Regan, A. Peterson, *Dynamic equations on time scales: A survey*, Journal of Computational and Applied Mathematics, 141 (2002), 1–26.

[6] U. Aksoy, T. Cuchta, S. Georgiev, Y.Y. Okur, *A Normal Distribution on Time Scales with Application*, Filomat, 36 (16) (2022), 5391–5404.

[7] N. Benkhettou, M. Bohner, A. Brito da Cruz, D. Torres, *A fractional calculus on arbitrary time scales: fractional differentiation and fractional integration*, Signal Processing, 107 (4) (2015), 230–237.

[8] M. Bohner, B. Kaymakcalan, *Opial inequalities on time scales*, Annales Polonici Mathematici, 77 (1) (2001), 11–20.

[9] M. Bohner, A. Peterson, *Dynamic equations on time scales*, Springer Science + Business Media, 2001.

[10] M. Bohner, O. M. Stanzhytskyi, A. O. Bratochkina, *Stochastic dynamic equations on general time scales*, Electronic Journal of Differential Equations, 57, (2013) 1–15.

[11] N.H. Du, N.T. Dieu, *The first attempt on the stochastic calculus on time scale*, Stochastic Analysis and Applications, 29 (6) (2011) 1057–1080.

[12] N.H. Du, N.T. Dieu, *Stochastic dynamic equations on time scales*, Acta Mathematica Vietnamica, 38 (2) (2013) 317–338.

[13] T. Fayyaz, N. Irshad, A. Khan, G. Rahman, G. Roqia, *Generalized integral inequalities on time scales*, Journal of Inequalities and Applications, 2016 (2016), Article ID: 235, 1–12.

[14] S. Hilger, *Makettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, PhD Thesis, Universität Würzburg, 1988.

[15] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264 (2014), 65–70.

[16] T. Matthews, *Probability theory on time scales and applications to finance and inequalities*, PhD Thesis, Missoury University of Science and Technology, 2011.

[17] V. Tarasov, *Lattice fractional calculus*, Applied Mathematics and Computation, 257 (2015), 12–33.

[18] D. Zhao, T. Li, *On conformable delta fractional calculus on time scales*, Journal of Mathematics and Computer Science, 16 (3) (2016), 324–335.

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